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Common Fixed Points for Occasionally Weakly Compatible Mappings in Cone Metric Spaces

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ABSTRACT
In this paper, we obtain some common fixed point theorems for occasionally weakly compatible mappings in cone metric spaces. Our results generalized and extend several existing fixed point theorems in the literature.
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I. INTRODUCTION AND PRELIMINARIES

In 1968 Kannan [7] was initiated study of fixed point theorems for a map satisfying a contractive condition that did not require continuity at each point. The notion of weakly commuting maps was initiated by Sessa [9].Jungck[5]gave the concept of of compatible maps and showed that weakly commuting mappings are compatible, but converse is not true. Junck further weakened the notion of compatibility by introducing weak compatibility. Pant [8] initiated the study of non compatible maps and introduced point wise R-weakly commutativity of mappings.Aamri and Moutawakil [2] introduced the (E.A) property and thus generalized the concept of non-compatible maps. Recently, Al-Thagafi and Shahzad [1] defined the concept of occasionally weakly compatible(owc) which is more general than the concept of weakly compatible maps. In 2007 Huang and Zhang [4] have generalized the concept of a metric space, replacing the set of real numbers by an ordered Banach space and obtained some fixed point theorems for mapping satisfying different contractive conditions. In this paper we prove common fixed point theorems for occasionally weakly compatible mappings in cone metric spaces. Our results extends the results of Guangxing Song et.al.[3] and S.L.Singh et.al.[10].

The following definitions are due to Huang and Zhang [4].

Definition 1.1. Let B be a real Banach Space and P a subset of B .The set P is called a cone if and only if:
(a). P is closed, non –empty and $P \neq \{0\}$;
(b). $a,b \in R$, $a,b \geq 0$, $x,y \in P$ implies $ax+by \in P$;
(c). $x \in P$ and $-x \in P$ implies $x = 0$.

Definition 1.2. Let P be a cone in a Banach Space B, define partial ordering '$\leq$' with respect to P by $x \leq y$ if and only if $y-x \in P$.We shall write $x<y$ to indicate $x \leq y$ but $x \neq y$ while $x<<y$ will stand for $y-x \in \text{Int } P$, where Int P denotes the interior of the set P. This Cone P is called an order cone.

Definition 1.3. Let B be a Banach Space and $P \subset B$ be an order cone .The order cone P is called normal if there exists $L>0$ such that for all $x,y \in B$,

$$0 \leq x \leq y \text{ implies } \|x\| \leq L\|y\|.$$

The least positive number L satisfying the above inequality is called the normal constant of P.

Definition 1.4. Let X be a nonempty set of B. Suppose that the map $d: X \times X \rightarrow B$ satisfies:
(d1).$0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
(d2).$d(x, y) = d(y, x)$ for all $x, y \in X$;
(d3).$d(x, y) \leq d(x, z) + d(y, z)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.
The concept of a cone metric space is more general than that of a metric space.

Example 1.5. ([4]). Let $B = R^2$, $P = \{(x , y) \in B$ such that : $x, y \geq 0\} \subset R^2$, $X = R$ and $d: X \times X \rightarrow B$ such that $d(x,y) = (|x - y|, \alpha|x - y|)$,where $\alpha \geq 0$ is a constant .Then (X, d) is a cone metric space.

Definition 1.6. Let (X, d) be a cone metric space. We say that $\{x_n\}$ is
(i) a Cauchy sequence if for every c in B with $c>>0$,there is N such that for all $n, m>N$, $d(x_n, x_m)<<c$;

(ii) convergent sequence if for any $c>>0$,there is an N such that for all $n>N$, $d(x_n, x) << c$, for some fixed x in X .We denote this $x_n \rightarrow x$ (as $\rightarrow \infty$) .

Lemma 1.1. Let (X, d) be a cone metric space, and let P be a normal cone with normal constant L.
Let $\{x_n\}$ be a sequence in X .Then
(i). $\{x_n\}$ converges to x if and only if $d(x_n ,x)\rightarrow 0$ $(n \rightarrow \infty)$.
(ii). $\{x_n\}$ is a Cauchy sequence if and only if $d(x_n , x_m)\rightarrow 0$ (as $n, m\rightarrow\infty$).

Definition 1.7. ([6]) Let X be a set and let f, g be two self-mappings of X. A point x in X is called a coincidence point of f and g iff fx = gx. We shall call w = fx = gx a point of coincidence point.

Definition 1.8. ([6]) Two self-maps f and g of a set X are occasionally weakly compatible(owc) iff there is a point x in X which is a coincidence point of f and g at which f and g commutate.

Lemma 1.2.([6]) Let X be a set f, g owc self-mappings of X. If f and g have a unique point of coincidence, w = fx = gx, then w is the unique common fixed point of f and g.

II. MAIN RESULTS

In this section, we prove some common fixed point theorems for occasionally weakly compatible in cone metric spaces.

Now we prove the following theorem.

Theorem (2.1): Let (X, d) be a cone metric space and P be a normal cone, let $a_i \geq 0$ (i=1,2, 3, 4, 5) be constants with $a_1 + a_2 +a_3 +a_4 +a_5<1$, and f, g self maps of X, f and g are occasionally weakly compatible(owc) , and satisfying

$$d(fx,fy)\leq a_1d(gx,gy)+a_2d(fx,gx)+a_3d(fy,gy) +a_4 d(gx,fy)+ a_5d(fx,gy) \text{ for all } x, y\in X. \quad \dots (1)$$

Then f and g have a unique common fixed point.

Proof: Since, f, g are owc, there exists a point $p\in X$ such that fp = gp, fgp = gfp.
We claim that fp is the unique common fixed point of f and g.
First we assert that fp is a fixed point of f.
For, if $ffp \neq fp$, then from (1), we get that

$$\begin{aligned}
d(fp,ffp) &\leq a_1d(gp, gfp)+a_2 d(fp,gp)+a_3 d(ffp,gfp) + a_4 d(gp,ffp)+ a_5d(fp,gfp) ,\\
&\leq a_1d(fp, ffp)+a_2 d(fp,fp)+a_3 d(ffp,gfp) +a_4 d(gp,ffp)+ a_5d(fp,ffp) ,\\
&\leq a_1d(fp,ffp)+0+a_3d(ffp,fp)+ a_3 d(fp,fgp) + a_4 d(gp,ffp)+ a_5d(fp,ffp) ,\\
&\leq (a_1+ a_3 + a_4 + a_5) d(fp,ffp) + a_3 d(fp,ffp),\\
&\leq (a_1 + 2a_3 + a_4 + a_5) d(fp,ffp) ,
\end{aligned}$$

a contradiction . Sine, $a_1 + a_2 +a_3 +a_4 +a_5<1$.
Hence, ffp = fp , fp is a fixed point of f .
And ffp = fgp = gfp = fp.
Thus, fp is a common fixed point of f and g.
Uniqueness, Suppose that $p,q\in X$ such that fp = gp = p and fq = gq =q and $p\neq q$..
Then (1) gives,

$$\begin{aligned}
d(p,q) &= d(fp,fq)\\
&\leq a_1d(gp,gq)+a_2d(fp,gp)+a_3d(fq,gq) +a_4 d(gp,fq)+ a_5d(fp,gq),\\
&\leq a_1d(p,q)+a_2 d(p,p)+a_3 d(q,q)+a_4 d(p,q) + a_5d(p,q),\\
&\leq a_1d(p,q)+0+0)+ a_4 d(p,q)+ a_5d(p,q),\\
&\leq (a_1+a_4+a_5)d(p,q)<d(p,q),\text{a contradiction.}
\end{aligned}$$

(Since, $a_1 + a_2 +a_3 +a_4 +a_5<1$)
Therefore, p = q.
Therefore, f and g have a unique common fixed point.

Theorem (2.2): Let (X, d) be a cone metric space and P be a normal cone. Suppose T and f are occasionally weakly compatible(owc), self- mappings of X, and satisfying the following conditions

$$d(Tx,Ty)\leq\varphi(g(x,y)) \text{ for all } x,y\in X \dots \quad (2)$$

where, $g(x,y) = d(fx,fy)+\gamma[d(fx, Tx)+ d(fy,Ty)]$, $0\leq\gamma\leq1$, and $\varphi: R_+ \rightarrow R_+$ continuous.
Then T and f have a unique common fixed point.

Proof: Since, T, f are owc, there exists a point $u\in X$ such that Tu = fu, Tfu = fTu.
We claim that fu is the unique common fixed point of T and f.
First we assert that fu is a fixed point of f.
For, if $ffu \neq fu$, then from (2), we get that

$$\begin{aligned}
d(fu, ffu) = d(Tu, Tfu) &\leq \varphi(g(u, fu))\\
&\leq \varphi(d(fu,ffu)+\gamma[d(fu, Tu)+ d(ffu,Tfu)]),\\
&\leq\varphi(d(fu, ffu)+\gamma[d(Tu, Tu)+ d(ffu,ffu)]),\\
&\leq\varphi(d(fu,ffu))< d(fu, ffu), \text{a contradiction.}
\end{aligned}$$

Therefore, ffu = fu, fu is a fixed point of f.
And ffu = Tfu = fTu = fu.
Thus, fu is a common fixed point of f and T.
Uniqueness, suppose that $u, v\in X$ such that fu = Tu = u and fv = Tv =v and $u\neq v$.
Then (2) gives,

$$\begin{aligned}
d(u,v) = d(Tu, Tv) &\leq \varphi(g(u, v))\\
&\leq \varphi(d(fu, fv)+\gamma[d(fu, Tu)+ d(fv,Tv)]) ,\\
&\leq \varphi(d(u, v))< d(u, v), \text{a contradiction.}
\end{aligned}$$

Therefore, u = v.
Therefore, T and f have a unique common fixed point.

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